MONOMIAL DEFORMATIONS OF CERTAIN HYPERSURFACES AND TWO HYPERGEOMETRIC FUNCTIONS.

KAZUAKI MIYATANI

ABSTRACT. The purpose of this article is to give an explicit description, in terms of hypergeometric functions over finite fields, of the zeta functions of certain smooth hypersurfaces that generalize the Dwork family. The key point here is that we count the number of rational points employing both the techniques of character sums and the theory of weights, which enables us to enlighten the calculation of the zeta function.

0. INTRODUCTION.

Hypergeometric functions in one complex variable with specific parameters appear as the periods of families of complex algebraic varieties. The most classical example is the function ${}_2F_1\left({1/2,1/2 \atop 1};\lambda\right)$ appearing in the period of the Legendre family of elliptic curves. Another example is the function ${}_nF_{n-1}\left({1/(n+1),2/(n+1),\dots,n/(n+1) \atop 1,\dots,1};\lambda^{-(n+1)}\right)$. As observed by Dwork [6], it appears in the period of the family of Calabi–Yau varieties defined by $T_1^{n+1}+\cdots+T_{n+1}^{n+1}-\lambda(n+1)T_1\dots T_{n+1}=0$ in the n-dimensional projective space; this family is now called the "Dwork family".

Turning to algebraic varieties over finite fields, the same hypergeometric functions, now regarded as power series, are still related to them. For example, a criterion for the existence of the unit-root (a Frobenius eigenvalue on the étale cohomology which is a p-adic unit) of the Legendre family, and for the unit-root itself if exists, can be described by using the hypergeometric series ${}_2F_1\left({1/2,1/2 \atop 1};\lambda\right)$ considered as a formal power series with p-adic coefficients; we also have an analogous theorem for the unit-root of the Dwork family [22]. We note that there are results of another type that describe the full zeta functions of certain Calabi–Yau threefolds in terms of hypergeometric series [8, Section 12].

Around 1990, Greene [12] and Katz [15] independently introduced an ℓ-adic version of hypergeometric functions using Gauss sums, which are called "hypergeometric functions over finite fields" or "Gaussian hypergeometric functions". These functions have a connection with algebraic varieties over finite fields analogously to the classical complex case as mentioned above. For example, the Frobenius traces of the first étale cohomology of each member of the Legendre family over $\mathbb{F}_q$ (with odd q) can be described by ${}_2F_1\left({\varphi_2,\varphi_2 \atop \varepsilon};\lambda\right)_{\mathbb{F}_q}$, where φ_2 (resp. ε) is the $\overline{\mathbb{Q}_\ell}$-valued character of order 2 (resp. the trivial character) on $\mathbb{F}_q^\times$. We have a similar theorem for the Dwork family, which relates the zeta

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function of (each member of) the Dwork family over $\mathbb{F}_q$ (with $q \equiv 1 \pmod{n+1}$) with ${}_nF_{n-1}\left({\varphi_{n+1},\varphi_{n+1}^2,\dots,\varphi_{n+1}^n \atop \varepsilon,\dots,\varepsilon}; \lambda^{-(n+1)}\right)_{\mathbb{F}_q}$, where φ_{n+1} is a $\overline{\mathbb{Q}_\ell}$-valued character of order $n+1$ on $\mathbb{F}_q^\times$ [16], [9], [10], [11].

So far, we have two strategies for proving such a relationship between hypergeometric functions over finite fields and the Frobenius traces of étale cohomology of algebraic varieties over finite fields. The first one is by counting rational points of the varieties via character sums. This strategy is classical and has been employed by many mathematicians; for example by Koike [17] for the Legendre family of elliptic curves, by Barman and Kalita [2] for more general families of curves and by Goutet [9], [10], [11] for the Dwork family. This method is powerful and we may get a quite explicit result with it. However, the method often requires complicated calculations concerning character sums; the success of the calculations highly depends on the shape (symmetry, for example) of the defining equations of the varieties. The second strategy relies on a geometric observation, which is remarkably employed by Katz in the case of the Dwork family and its generalizations [16]. Here the first step is to decompose the cohomology sheaf by using a finite abelian group acting on this variety, and then to relate each component with an ℓ-adic hypergeometric sheaf. This method is not only natural but has an advantage that we may avoid intricate calculations. However, the argument is not done over $\mathbb{F}_q$ but over an algebraic closure of $\mathbb{F}_q$; as a result, we may only relate each component with an ℓ-adic hypergeometric sheaf modulo the tensorization of a character sheaf [16, Theorem 5.3, Question 5.5]. (Although it does not concern hypergeometric functions over finite fields but just Jacobi sums, it is worth remarking that another motivic argument is available [8, Section 8] for determining the zeta functions for some K3 surfaces.)

In this article, we provide more general examples of families of hypersurfaces whose zeta functions are related to hypergeometric functions over finite fields, employing a method in which these two strategies are interwined. To be more precise, our method basically follows the first strategy (counting rational points via character sums), but we do not forget the fact that the hypergeometric functions over finite fields have a geometric (or ℓ-adic) nature, and use this fact to eliminate the factors with improper weights. This allows us to avoid some messy calculations of character sums, and as a result, we obtain a description of the zeta functions of much more general hypersurfaces in terms of hypergeometric functions over finite fields.

The hypersurfaces on which we work are the monomial deformations of hypersurfaces of degree $n+1$ in $\mathbb{P}^n_{\mathbb{F}_q}$ obtained by adding the sum of $n+1$ monomials. Unlike most of the previous results, we do not require the hypersurfaces we start with are diagonal ones. Explicitly, we consider the smooth hypersurface X_λ, which is defined in $\mathbb{P}^n_{\mathbb{F}_q}$ by

$$F_\lambda(T_1,\dots,T_{n+1}) = c_1T^{a_1} + \cdots + c_{n+1}T^{a_{n+1}} - \lambda T_1\dots T_{n+1},$$

where $c_1,\dots,c_{n+1} \in \mathbb{F}_q^\times$, $a_i = (a_{1i},\dots,a_{n+1,i}) \in \mathbb{Z}_{\geq 0}^{n+1}$, and $T^{a_i} = T_1^{a_{1i}}\dots T_{n+1}^{a_{n+1,i}}$. We assume that X_0 is smooth, and we view X_λ as a monomial deformation of X_0 (moreover, we assume that $q-1$ is divisible by some numbers defined by F_0; see Subsection 3.1).

Finally, besides this main theme, we note that this hypersurface X_λ also has a relationship with the usual hypergeometric series viewed as having p-adic coefficients as in the classical examples mentioned in the beginning of the introduction. We also investigate and prove this fact in this article.

We conclude this introduction by explaining the structure of this article.

Section 1 is devoted to the foundation of the hypergeometric functions over finite fields. In Subsection 1.1, we recall the definition and basic properties of the hypergeometric functions over finite fields. In Subsection 1.2, we introduce a notion of "q-Weil functions", which describes the property of "coming from an ℓ-adic sheaf", and restate the existence of ℓ-adic hypergeometric sheaves in this terminology.

In Section 2, we start the study of the zeta function of the hypersurface X_λ. Subsection 2.1 is devoted to introducing some data which are used to describe the hypergeometric functions related to X_λ. Subsection 2.2 is a preparation for Subsection 2.3, in which we describe a criterion for the existence of the unit-root. Moreover, if the unit-root exists, we obtain a description of it in terms of the usual hypergeometric series (in p-adic coefficients).

Section 3 is the main part of this article, which is devoted to the calculation of the zeta function of X_λ. Subsection 3.1 is devoted to stating explicit assumptions of the main theorem and introducing the other data which we need for our calculation of the zeta function. In Subsection 3.2, we state the main theorem, which describes the zeta function of X_λ in terms of hypergeometric functions over finite fields. We then reduce the proof to a direct calculation of character sums. We complete the proof in Subsection 3.3.

Notations. Throughout this article, we fix a prime number p, a power q of p and an integer n greater than or equal to 2. The finite field with q^r elements (for a positive integer r) is denoted by $\mathbb{F}_{q^r}$.

We denote the group of characters $\mathbb{F}_q^\times \to \overline{\mathbb{Q}}^\times$ by $\widehat{\mathbb{F}_q^\times}$, and the trivial character by ε. A character $\chi \in \widehat{\mathbb{F}_q^\times}$ is also considered as a function $\mathbb{F}_q \to \overline{\mathbb{Q}}$ by setting $\chi(0) = 0$.

By default, an element in A^m for an abelian group A and a natural number m is considered as a column vector. For such a vector $b = {}^{\mathrm{t}}(b_1, \dots, b_m)$, we write $|b| := b_1 + \cdots + b_m$.

Finally, *throughout this article, we fix a non-trivial additive character* $\theta\colon \mathbb{F}_q \to \overline{\mathbb{Q}_\ell}$.

1. Preliminaries on Hypergeometric Functions.

1.1. **Definitions.** In this subsection, we review the hypergeometric functions over finite fields. First, let us recall the usual hypergeometric series.

Definition 1.1. *For $2n+1$ rational numbers $A_1, \dots, A_{n+1}, B_1, \dots, B_n$ such that none of B_j's belong to $\mathbb{Z}_{\le 0}$, we define the hypergeometric series*

$$ {}_{n+1}F_n\left({A_1, \dots, A_{n+1} \atop B_1, \dots, B_n} ; x \right) := \sum_{k=0}^{\infty} \frac{(A_1)_k \dots (A_{n+1})_k}{(B_1)_k \dots (B_n)_k (1)_k} x^k. $$

Here, for a number c and a natural number k, the Pochhammer symbol $(c)_k$ is defined to be the product $c(c+1)\dots(c+k-1)$, i.e., $(c)_k := \Gamma(c+k)/\Gamma(c)$ by using the Gamma function.

Now, we introduce the hypergeometric function over finite fields; recall from the last paragraph of Notations that we fixed a non-trivial additive character $\theta\colon \mathbb{F}_q \to \overline{\mathbb{Q}}^\times$.

Definition 1.2. *Let $A_1, \dots, A_{n+1}$ and $B_1, \dots, B_{n+1}$ be characters of $\mathbb{F}_q^\times$. We define the* hypergeometric function over $\mathbb{F}_q$ *with these parameters as*

$$ {}_{n+1}\widetilde{F}_{n+1}\left({A_1, \dots, A_{n+1} \atop B_1, \dots, B_{n+1}} ; x \right)_{\mathbb{F}_q} = \frac{1}{q-1} \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} \prod_{i=1}^{n+1} \frac{G(A_i\chi)}{G(A_i)} \frac{G(\overline{B_i\chi})}{G(\overline{B_i})} \chi(-1)^{n+1} \chi(x); $$

here, $G(\mu)$ *(for a multiplicative character* μ*) is the Gauss sum with respect to the fixed* θ*, that is,* $G(\mu) := \sum_{x\in\mathbb{F}_q^\times}\theta(x)\mu(x)$*. Moreover, we define*

$$ {}_{n+1}F_n\left(\begin{matrix}A_1,\dots,A_{n+1}\\ B_1,\dots,B_n\end{matrix};x\right)_{\mathbb{F}_q} = {}_{n+1}\widetilde{F}_{n+1}\left(\begin{matrix}A_1,\dots,A_{n+1}\\ B_1,\dots,B_n,\varepsilon\end{matrix};x\right)_{\mathbb{F}_q}. $$

Remark 1.3. *In the literature, there exist a lot of variants of hypergeometric functions over finite fields besides the original definitions by Greene* [12] *and Katz* [15]*. Our function* ${}_{n+1}F_n\left(\begin{smallmatrix}A_1,\dots,A_{n+1}\\ B_1,\dots,B_n\end{smallmatrix};x\right)_{\mathbb{F}_q}$ *coincides with the definition by McCarthy* [18, Definition 1.4].

An important feature of the hypergeometric functions over finite fields is a geometric interpretation. In fact, Katz [15, 8.2] constructs a $\overline{\mathbb{Q}_\ell}$-sheaf $\mathscr{H}(!,\theta;A_1,\dots,A_{n+1};B_1,\dots,B_{n+1})$ on $\mathbb{G}_{\mathrm{m}}$, which is smooth on $\mathbb{G}_{\mathrm{m}}\setminus\{1\}$ and pure of weight $2n+1$ if $\{A_1,\dots,A_{n+1}\}\cap\{B_1,\dots,B_{n+1}\}=\emptyset$ [15, Theorem 8.4.2 (4)]. We may show, in the same way as [18, Proposition 2.6], that the trace of Frobenius action on the sheaf at $x\in\mathbb{F}_q^\times$ is equal to

$$ \prod_{i=1}^{n+1} G(A_i)G(B_i)\cdot {}_{n+1}\widetilde{F}_{n+1}\left(\begin{matrix}A_1,\dots,A_{n+1}\\ B_1,\dots,B_{n+1}\end{matrix};x^{-1}\right)_{\mathbb{F}_q}. $$

1.2. q**-Weil functions.** In this subsection, we introduce a concept of q-Weil functions, and prove that the Gauss sums and the hypergeometric functions over finite fields give examples of such functions. Recall that an algebraic number α is said to be a *q-Weil number of weight k*, where k is an integer, if for all embeddings $\iota\colon\overline{\mathbb{Q}}\to\mathbb{C}$, the absolute value of $\iota(\alpha)$ equals $q^{k/2}$.

Definition 1.4. *Let* $f\colon\mathbb{Z}_{>0}\to\overline{\mathbb{Q}}$ *be a function and* k *an integer. We say that* f *is a* pure q-Weil function of weight k *if there exist some* q*-Weil numbers* $\alpha_1,\dots,\alpha_m$ *of weight* k *satisfying*

$$ f(r)=\sum_{i=1}^m \alpha_i^r \quad (\forall r\in\mathbb{Z}_{>0}). $$

We say that f *is a* (mixed) q-Weil function *if there exist some pure* q*-Weil functions* f_i *of weight* k_i $(i=1,\dots,m)$ *and signs* $\varepsilon_1,\dots,\varepsilon_m\in\{1,-1\}$ *satisfying*

$$ f=\sum_{i=1}^m \varepsilon_i f_i. $$

f *is a* (mixed) q-Weil function of weight $\le k$ (resp. of weight $\ge k$, of weight $\ne k$) *if all* k_i*'s can be taken to be* $\le k$ (resp. $\ge k$, $\ne k$).

The property of being a q-Weil function can be restated in terms of the associated zeta function defined below.

Definition 1.5. *The zeta function* $\zeta(f)(T)$ *of a function* $f\colon\mathbb{Z}_{>0}\to\overline{\mathbb{Q}}$ *is the formal power series*

$$ \zeta(f)(T)=\exp\left(-\sum_{r=1}^\infty f(r)\frac{T^r}{r}\right)\in 1+T\overline{\mathbb{Q}}[[T]]. $$

The following proposition follows from a standard calculation.

Proposition 1.6. *A non-zero function* $f\colon\mathbb{Z}_{>0}\to\overline{\mathbb{Q}}$ *is a pure* q*-Weil function of weight* k *if and only if its zeta function* $\zeta(f)(T)$ *is a polynomial in* $\overline{\mathbb{Q}}[T]$ *all of whose reciprocal roots are* q*-Weil numbers of weight* k*.*

Corollary 1.7. *For each non-zero q-Weil function f, there uniquely exist mutually different numbers $k_1,\dots,k_r$ and non-zero q-Weil functions f_i of weight k_i for each i such that $f=\sum f_i$.*

To state explicitly the q-Weil property of Gauss sums and hypergeometric functions, we introduce the following notation.

Notation 1.8. *Let r be a positive integer. Then, we denote the non-trivial additive character $\theta\circ\mathrm{Tr}_{\mathbb{F}_{q^r}/\mathbb{F}_q}$ on $\mathbb{F}_{q^r}$ by θ_r. Similarly, for each character χ on $\mathbb{F}_q^\times$, the character $\chi\circ\mathrm{Norm}_{\mathbb{F}_{q^r}/\mathbb{F}_q}$ on $\mathbb{F}_{q^r}^\times$ is denoted by χ_r. The Gauss sum of a character χ' on $\mathbb{F}_{q^r}^\times$ with respect to θ_r is denoted by $G(\chi')$.*

Moreover, for $2(n+1)$ characters $A_1,\dots,A_{n+1},B_1,\dots,B_{n+1}$ of $\mathbb{F}_q^\times$, we define ${}_{n+1}\widetilde{F}_{n+1}\left(\begin{smallmatrix}A_1,\dots,A_{n+1}\\B_1,\dots,B_{n+1}\end{smallmatrix};x\right)_{\mathbb{F}_{q^r}}$ to be

$$\frac{1}{q^r-1}\sum_{\chi\in\widehat{\mathbb{F}_{q^r}^\times}}\prod_{i=1}^{n+1}\frac{G(A_{i,r}\chi)}{G(A_{i,r})}\frac{G(\overline{B_{i,r}\chi})}{G(\overline{B_{i,r}})}\chi(-1)^{n+1}\chi(x);$$

this coincides with the function ${}_{n+1}\widetilde{F}_{n+1}\left(\begin{smallmatrix}A_{1,r},\dots,A_{n+1,r}\\B_{1,r},\dots,B_{n+1,r}\end{smallmatrix};x\right)_{\mathbb{F}_{q^r}}$ in Definition 1.2, where the non-trivial additive character on $\mathbb{F}_{q^r}$ needed to define it is θ_r.

Finally, the function ${}_{n+1}F_n\left(\begin{smallmatrix}A_1,\dots,A_{n+1}\\B_1,\dots,B_n\end{smallmatrix};x\right)_{\mathbb{F}_{q^r}}$ is defined to be ${}_{n+1}\widetilde{F}_{n+1}\left(\begin{smallmatrix}A_1,\dots,A_{n+1}\\B_1,\dots,B_n,\varepsilon\end{smallmatrix};x\right)_{\mathbb{F}_{q^r}}$.

Proposition 1.9. *Let χ be a character on $\mathbb{F}_q^\times$. Then, for any positive integer r,*

$$-G(\chi_r)=\left(-G(\chi)\right)^r.$$

In particular, the function $r\mapsto -G(\chi_r)$ is a pure q-Weil function, whose weight is 1 if χ is non-trivial and is 0 if χ is trivial.

Proof. The first assertion is a result of Davenport and Hasse [3, (0.8)]. The second assertion follows from the first assertion and from the standard fact that $-G(\chi_r)$ itself is a q^r-Weil number of weight 1 if χ_r is non-trivial and of weight 0 if χ_r is trivial. □

Now, let us state the q-Weil property of the hypergeometric functions over finite fields without common characters in upper and lower parameters.

Proposition 1.10. *Let $A_1,\dots,A_{n+1},B_1,\dots,B_{n+1}$ be characters on $\mathbb{F}_q^\times$, and assume that $\{A_1,\dots,A_{n+1}\}\cap\{B_1,\dots,B_{n+1}\}=\emptyset$. Let m be the number of trivial characters among A_i's and B_i's. Then, for all $x\in\mathbb{F}_q^\times\setminus\{1\}$, the function*

$$r\mapsto {}_{n+1}\widetilde{F}_{n+1}\left(\begin{matrix}A_1,\dots,A_{n+1}\\B_1,\dots,B_{n+1}\end{matrix};x\right)_{\mathbb{F}_{q^r}}$$

is a pure q-Weil function of weight $m-1$.

Proof. We use the notation in the paragraph after Remark 1.3. Since the base extension of the $\overline{\mathbb{Q}_\ell}$-sheaf $\mathscr{H}(!,\theta;A_1,\dots,A_{n+1};B_1,\dots,B_{n+1})$ to $\mathbb{F}_{q^r}$ is isomorphic to $\mathscr{H}(!,\theta_r;A_{1,r},\dots,A_{n+1,r};B_{1,r},\dots,B_{n+1,r})$ [15, (8.2.6)], the function

$$r\mapsto\prod_{i=1}^{n+1}G(A_{i,r})G(B_{i,r})\cdot{}_{n+1}\widetilde{F}_{n+1}\left(\begin{matrix}A_1,\dots,A_{n+1}\\B_1,\dots,B_{n+1}\end{matrix};x^{-1}\right)_{\mathbb{F}_{q^r}}$$

is a pure q-Weil function of weight $2n+1$. In turn, Proposition 1.9 shows that, if we denote $f(r):=\prod_{i=1}^{n+1}G(A_{i,r})G(B_{i,r})$, then the function $r\mapsto f(r)$ is a pure q-Weil function of weight $2(n+1)-m$ and satisfies $f(r)=f(1)^r$. This shows the proposition. □

In the classical hypergeometric functions, we may cancel common numbers in upper and lower parameters without any change to the functions themselves. For the hypergeometric functions over finite fields, however, the situation is different.

Definition 1.11. *Let $A_1,\dots,A_{n+1},B_1,\dots,B_{n+1}$ be characters on $\mathbb{F}_q^\times$. By changing indices, we, without loss of generality, assume that $\{A_1,\dots,A_{n'+1}\}$ and $\{B_1,\dots,B_{n'+1}\}$ have an empty intersection and that $\{A_{n'+2},\dots,A_{n+1}\}$ and $\{B_{n'+2},\dots,B_{n+1}\}$ coincide as multi-sets. Then, for each positive integer r, we define the* hypergeometric function with reduced parameters over $\mathbb{F}_{q^r}$ *by*

$$_\bullet\widetilde{F}_\bullet\mathrm{Red}\left(\begin{matrix}A_1,\dots,A_{n+1}\\B_1,\dots,B_{n+1}\end{matrix};x\right)_{\mathbb{F}_{q^r}}:={}_{n'+1}\widetilde{F}_{n'+1}\left(\begin{matrix}A_1,\dots,A_{n'+1}\\B_1,\dots,B_{n'+1}\end{matrix};x\right)_{\mathbb{F}_{q^r}}.$$

Proposition 1.12. *Let $A_1,\dots,A_{n+1},B_1,\dots,B_{n+1}$ be characters on $\mathbb{F}_q^\times$, and let m be the number of trivial characters among B_i's. Fix an element x of $\mathbb{F}_q^\times$.*

(i) *If all A_i's are non-trivial, then the function*

$$r\mapsto{}_{n+1}\widetilde{F}_{n+1}\left(\begin{matrix}A_1,\dots,A_{n+1}\\B_1,\dots,B_{n+1}\end{matrix};x\right)_{\mathbb{F}_{q^r}}-{}_\bullet\widetilde{F}_\bullet\mathrm{Red}\left(\begin{matrix}A_1,\dots,A_{n+1}\\B_1,\dots,B_{n+1}\end{matrix};x\right)_{\mathbb{F}_{q^r}}$$

is a q-Weil function of weight $\le m-2$.

(ii) *If exactly one of A_i's is trivial, and if at least one of B_i's is trivial, then the function*

$$r\mapsto{}_{n+1}\widetilde{F}_{n+1}\left(\begin{matrix}A_1,\dots,A_{n+1}\\B_1,\dots,B_{n+1}\end{matrix};x\right)_{\mathbb{F}_{q^r}}-q^r{}_\bullet\widetilde{F}_\bullet\mathrm{Red}\left(\begin{matrix}A_1,\dots,A_{n+1}\\B_1,\dots,B_{n+1}\end{matrix};x\right)_{\mathbb{F}_{q^r}}$$

is a q-Weil function of weight $\le m-1$.

In particular, in both cases, the given function is a q-Weil function if $x\ne1$. Moreover, it is of weight strictly less than the weight of the second term of the function.

Proof. To prove (i), it suffices to show that, for non-trivial characters $A_1,\dots,A_n,C$ and arbitrary characters $B_1,\dots,B_n$, the function

$$r\mapsto{}_{n+1}\widetilde{F}_{n+1}\left(\begin{matrix}A_1,\dots,A_n,C\\B_1,\dots,B_n,C\end{matrix};x\right)_{\mathbb{F}_{q^r}}-{}_n\widetilde{F}_n\left(\begin{matrix}A_1,\dots,A_n\\B_1,\dots,B_n\end{matrix};x\right)_{\mathbb{F}_{q^r}}$$

is a q-Weil function of weight $\le m-2$.

By definition, ${}_{n+1}\widetilde{F}_{n+1}\left(\begin{smallmatrix}A_1,\dots,A_n,C\\B_1,\dots,B_n,C\end{smallmatrix};x\right)_{\mathbb{F}_q}$ equals

$$\frac{1}{q-1}\sum_{\chi\in\widehat{\mathbb{F}_q^\times}}\frac{G(A_1\chi)\dots G(A_n\chi)G(C\chi)}{G(A_1)\dots G(A_n)G(C)}\frac{G(\overline{B_1\chi})\dots G(\overline{B_n\chi})G(\overline{C\chi})}{G(\overline{B_1})\dots G(\overline{B_n})G(\overline{C})}\chi((-1)^{n+1})\chi(x).\tag{1}$$

Because $G(\mu)G(\overline{\mu})$ equals $q\mu(-1)$ if μ is a non-trivial character on $\mathbb{F}_q$ and equals $q\mu(-1)-(q-1)$ if μ is the trivial character,

$$\frac{G(C\chi)G(\overline{C\chi})}{G(C)G(\overline{C})}=\begin{cases}\dfrac{qC\chi(-1)}{qC(-1)}=\chi(-1) & \text{if }\chi\ne C^{-1},\\[2ex] \dfrac{qC\chi(-1)-(q-1)}{qC(-1)}=\chi(-1)-\dfrac{q-1}{q}C(-1) & \text{if }\chi=C^{-1},\end{cases}$$

which shows that (1) equals

$$\frac{1}{q-1}\sum_{\chi\in\widehat{\mathbb{F}_q^\times}}\frac{G(A_1\chi)\dots G(A_n\chi)}{G(A_1)\dots G(A_n)}\frac{G(\overline{B_1\chi})\dots G(\overline{B_n\chi})}{G(\overline{B_1})\dots G(\overline{B_n})}\chi((-1)^n)\chi(x)$$
$$-\frac{1}{q}\frac{G(A_1C^{-1})\dots G(A_nC^{-1})}{G(A_1)\dots G(A_n)}\frac{G(\overline{B_1C^{-1}})\dots G(\overline{B_nC^{-1}})}{G(\overline{B_1})\dots G(\overline{B_n})}C((-1)^n)C^{-1}(x).$$

Its first term equals ${}_n\widetilde{F}_n\left({A_1,\dots,A_n \atop B_1,\dots,B_n};x\right)_{\mathbb{F}_q}$. By Proposition 1.9, the second term with q replaced by q^r for various r gives a q-Weil function of weight $\leq 2n-(2n-m)-2=m-2$.

To prove (ii), we show that, for non-trivial characters $A_1\dots,A_n$, arbitrary characters $B_1,\dots,B_n$ and the trivial character C, the function

$$r\mapsto {}_{n+1}\widetilde{F}_{n+1}\left({A_1,\dots,A_n,C \atop B_1,\dots,B_n,C};x\right)_{\mathbb{F}_{q^r}}-q^r\,{}_n\widetilde{F}_n\left({A_1,\dots,A_n \atop B_1,\dots,B_n};x\right)_{\mathbb{F}_{q^r}}$$

is a q-Weil function of weight $\leq m-2$, and then it suffices to use (i). This is proved in the same way as in (i) by using the following equation for the trivial C:

$$\frac{G(C\chi)G(\overline{C\chi})}{G(C)G(\overline{C})}=\begin{cases}q\chi(-1) & \text{if } \chi\neq\varepsilon,\\ q\chi(-1)-(q-1)C(-1) & \text{if } \chi=\varepsilon.\end{cases}$$

The last paragraph of the statement follows from Proposition 1.10. □

2. Monomial Deformations and p-adic Hypergeometric Series.

2.1. **Families of hypersurfaces considered.** In this subsection, we introduce families of hypersurfaces on which we work, and set some notations concerning them. Let X_0 be the *smooth* hypersurface in $\mathbb{P}^n_{\mathbb{F}_q}$ defined by the polynomial

$$F_0(T)=c_1T^{a_1}+\dots+c_{n+1}T^{a_{n+1}}\in\mathbb{F}_q[T_1,\dots,T_{n+1}],$$

where $c_1,\dots,c_{n+1}\in\mathbb{F}_q^\times$ and where $a_1,\dots,a_{n+1}\in\mathbb{Z}_{\geq 0}^{n+1}$ with $|a_i|=n+1$ $(i=1,\dots,n+1)$, none of a_i's being equal to ${}^{t}(1,1,\dots,1)$. Here, if $a_i={}^{t}(a_{1i},\dots,a_{n+1,i})$, the notation T^{a_i} means the monomial $T_1^{a_{1i}}\dots T_{n+1}^{a_{n+1,i}}$.

The theme of this article is investigating the monomial deformation X_λ of X_0 defined by the polynomial

$$F_\lambda(T)=c_1T^{a_1}+\dots+c_{n+1}T^{a_{n+1}}-\lambda T_1\dots T_{n+1},$$

where λ moves in $\mathbb{F}_q^\times$. We will mainly restrict our attention to those λ such that X_λ is smooth.

By the following proposition, we may and do assume (by changing the indices if necessary) that each diagonal entry of the exponent matrix A equals $n+1$ or n.

Proposition 2.1. *Let A be the matrix $A=(a_1,\dots,a_{n+1})$. Then, after a suitable change of indices of a_i's, each diagonal entry of A equals $n+1$ or n, the other entries are either* 0 *or* 1*, and moreover there exists at most one* 1 *in each row.*

Proof. First, let us prove the first property. It suffices to show that, for each fixed $i=1,\dots,n+1$, there exists an index j with $a_{ij}\geq n$. Assume that there exists an i that satisfies

$a_{ij} \le n-1$ ($j = 1, \dots, n+1$). Fix such an i, and let P_i denote the point $[0 : \cdots : 0 : 1 : 0 : \cdots : 0]$ in $\mathbb{P}^n_{\mathbb{F}_q}$, where 1 sits in the i-th entry. Then, for each $j, k \in \{1, \dots, n+1\}$, we have

$$\frac{\partial T^{a_j}}{\partial T_k} = \begin{cases} a_{kj} T^{a_j - e_k} & \text{if } a_{kj} \ge 1, \\ 0 & \text{if } a_{kj} = 0, \end{cases}$$

where e_k denotes the vector ${}^{\mathrm{t}}(0, \dots, 0, 1, 0, \dots, 0)$ with 1 sitting in the k-th entry. The value of this partial derivative at P_i is zero unless $T^{a_j - e_k} = T_i^n$, which is impossible because of the assumption on i. Therefore, we have shown that

$$\frac{\partial F_0}{\partial T_k}\big(P_i\big) = 0 \quad (k = 1, \dots, n+1) \quad \text{and} \quad F_0\big(P_i\big) = 0,$$

and consequently P_i is a singular point of X_0, which contradicts the smoothness of X_0.

The second property follows from the first property and from the assumption that $|a_i| = n+1$ for all i.

In order to prove the third property, we assume the contrary. Then, after a change of coordinates, we may assume that $a_{12} = a_{13} = 1$, that is, $T^{a_2} = T_1 T_2^n$ and $T^{a_3} = T_1 T_3^n$. For an element x of $\overline{\mathbb{F}_q}$ that satisfies $c_2 + c_3 x^n = 0$, the point $P = [0 : 1 : x : 0 : \cdots : 0]$, which is actually an $\overline{\mathbb{F}_q}$-rational point of X_0, gives a singular point of X_0. In fact, the choice of x shows that $\partial F_0 / \partial T_1\big(P\big) = 0$, and it is straightforward to check that $\partial F_0 / \partial T_i\big(P\big) = 0$ for all $i \ge 2$; thus we have derived a contradiction. □

In order to write the parameters and the input of the hypergeometric functions, we introduce some notation.

Proposition 2.2. *The kernel of the homomorphism $\mathbb{Z}^{n+1} \to \mathbb{Z}^{n+1}$ defined by the matrix $A' = \big(a_{ij} - 1\big)_{i,j=1,\dots,n+1}$ is free of rank one and generated by a (uniquely determined) vector ${}^{\mathrm{t}}(\alpha_1, \dots, \alpha_{n+1})$ with all $\alpha_i > 0$.*

Proof. The matrix A' is not invertible since $(1, 1, \dots, 1) A' = 0$.

It, therefore, suffices to show that every entries of a non-zero vector ${}^{\mathrm{t}}(x_1, \dots, x_{n+1})$ in the kernel have the same sign. By multiplying every entries by -1 if necessary, we may assume that at least two entries are non-negative. After a suitable change of indices, we may assume that $x_1 \ge x_2 \ge \cdots \ge x_{n+1}$; consequently, $x_1 > 0$ and $x_2 \ge 0$.

Now, if an index $i \in \{1, \dots, n\}$ satisfies $x_i > 0$ and $x_{i+1} \le 0$, then

$$(a_{11} - 1) x_1 + \sum_{j=i+1}^{n+1} (a_{1j} - 1) x_j = \sum_{j=2}^{i} -(a_{1j} - 1) x_j.$$

The left-hand side is greater than or equal to $(n-1)x_1$ since $(a_{1j} - 1)x_j \ge 0$ for $j = i+1, \dots, n+1$. Now, $-(a_{1j} - 1)$ being 0 or 1 for $j \ne 1$, the right-hand side is less than or equal to $(i-1)x_1$. This shows that $i = n$, which implies the contradiction

$$0 = \sum_{j=1}^{n} (a_{n+1,j} - 1) x_j + (a_{n+1,n+1} - 1) x_{n+1} < 0$$

because $a_{n+1,j} = 0$ for at least one $j \in \{1, \dots, n\}$. □

We fix the notation $\alpha_1, \dots, \alpha_{n+1}$ henceforth. The sum $\sum_{i=1}^{n+1} \alpha_i$ is denoted by α. Throughout this section, we always assume the following condition.

Assumption 2.3. *q is relatively prime to all α_i's and to α.*

Definition 2.4. *We define an element* $C \in \mathbb{F}_q^\times$ *by*

$$C = \alpha^\alpha \frac{c_1^{\alpha_1}}{\alpha_1^{\alpha_1}} \cdots \frac{c_{n+1}^{\alpha_{n+1}}}{\alpha_{n+1}^{\alpha_{n+1}}}.$$

Example 2.5. *(i) In the case of the Dwork family, that is, if* $F_0(T) = T_1^{n+1} + \cdots + T_{n+1}^{n+1}$*, then* ${}^{\mathrm{t}}(\alpha_1, \alpha_2, \dots, \alpha_{n+1}) = {}^{\mathrm{t}}(1, 1, \dots, 1)$*. Since* $\alpha = n+1$*, we have* $C = (n+1)^{n+1}$.

(ii) Let us consider the following example discussed by Yu and Yui [23, (4.8.1)]*:* $n = 3$ *and* $F_0(T) = T_1^4 + T_1 T_2^3 + T_3^4 + T_4^4$*. Then, we have* ${}^{\mathrm{t}}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = {}^{\mathrm{t}}(2, 4, 3, 3)$*. Since* $\alpha = 12$*, we have* $C = 2^{14} \cdot 3^6$.

2.2. **Formal group laws.** In this subsection, we recall some facts on formal group laws and a special case of Artin–Mazur functors, and apply them to our case. For the basic language of formal group laws, the reader may consult Hazewinkel's book [13].

Let R be a (commutative unitary) ring. A one-dimensional commutative formal group law over R, which we simply say "a formal group law over R" in this article, is a formal power series $G(X, Y) \in R[[X, Y]]$ that satisfies the conditions corresponding to group axioms [13, 1.1]. A logarithm of the formal group law G is a formal power series $l(\tau) \in R[[\tau]]$ that satisfies $l(\tau) \equiv \tau \pmod{\deg 2}$ and $G(X, Y) = l^{-1}\big(l(X) + l(Y)\big)$. Equivalently, a logarithm of G is a strict isomorphism of G to the additive group $\widehat{\mathbb{G}_{\mathrm{a}}}$ defined by $\widehat{\mathbb{G}_{\mathrm{a}}}(X, Y) = X + Y$. A logarithm of a formal group law is unique if the ring R is of characteristic zero, and it exists if R contains $\mathbb{Q}$. If R is of characteristic zero, we also say that $l(\tau) \in R \otimes_{\mathbb{Z}} \mathbb{Q}[[\tau]]$ is a logarithm of the formal group $G(X, Y)$ over R, if it is the logarithm of $G(X, Y)$ viewed as a formal group law over $R \otimes_{\mathbb{Z}} \mathbb{Q}$.

For a formal group law $G(X, Y)$ over a ring R and a positive integer m, define a formal power series $[m]_G(X)$ inductively by

$$[1]_G(X) = X, \qquad [m]_G(X) = G\big(X, [m-1]_G(X)\big) \quad (m \geq 2).$$

Assume that R is a field of characteristic p. If $[p]_G(X)$ is non-zero, then the lowest term of $[p]_G(X)$ is of degree p^h for a positive integer h. We call this h the *height* of $G(X, Y)$; if $[p]_G(X)$ is zero, the height is defined to be infinity.

Lemma 2.6. *Let R be a ring of characteristic zero whose reduction modulo p is a field k. Let $G(X, Y)$ be a formal group law over R with logarithm of the form* $l(\tau) = \sum_{s=0}^\infty a_s \tau^{p^s}/p^s$ $(a_s \in R,\ a_0 = 1)$*. Then, the formal group law* $\overline{G}(X, Y) := G(X, Y) \mod p$ *over k is of height one if and only if* $a_1 \not\equiv 0 \pmod{p}$.

Proof. Let u denote the coefficient of X^p in the formal power series $[p]_G(X)$. Then, $[p]_G(X)$ is of the form

$$[p]_G(X) = pX + v_2 X^2 + \cdots + v_{p-1} X^{p-1} + u X^p + \big(\text{degree } \geq p+1\big) \tag{2}$$

where $v_2, \dots, v_{p-1} \equiv 0 \pmod{p}$; the formal group law $\overline{G}$ is of height one if and only if $u \not\equiv 0 \pmod{p}$. The definition of the logarithm implies the equation

$$l\big([p]_G(X)\big) = pl\big(X\big) = pX + a_1 X^p + a_2 \frac{X^{p^2}}{p} + \dots.$$

By substituting (2), the coefficient of X^p in the left-hand side equals u modulo p, and the lemma follows. ∎

Next, we briefly recall the theory of formal groups. Let R be a ring and let $\mathbf{NilAlg}_R$ denote the category of nilpotent R-algebras. For a natural number n, the functor $\widehat{\mathbb{A}}^n\colon \mathbf{NilAlg}_R \to \mathbf{Set}$ is defined by $\widehat{\mathbb{A}}^n(N) = N^n$. We say that a functor $G\colon \mathbf{NilAlg}_R \to \mathbf{Ab}$ is an *n-dimensional formal group* if its composition with the forgetful functor $\mathbf{Ab} \to \mathbf{Set}$ is isomorphic to $\widehat{\mathbb{A}}^n$. The category of (one-dimensional commutative) formal group laws and the category of one-dimensional formal groups are equivalent.

Let X be a scheme over a perfect field k of characteristic p, and i a natural number. Then, we define the *Artin–Mazur functor* by

$$H^i\big(X, \widehat{\mathbb{G}_{\mathrm{m}}}\big)\colon \mathbf{NilAlg}_R \to \mathbf{Ab}; \quad A \mapsto H^i\big(X, \widehat{\mathbb{G}_{\mathrm{m}}}(\mathscr{O}_X \otimes_R A)\big).$$

Although this may or may not be a formal group, it actually is a formal group in the case of our interest. In fact, the Artin–Mazur functor is deeply related to the Witt cohomology of a variety over k; if X is a complete intersection of dimension $d \geq 2$, then $H^d\big(X, \widehat{\mathbb{G}_{\mathrm{m}}}\big)$ is a formal group, and the Cartier–Dieudonné module of this formal group is isomorphic to $H^d\big(X, W\mathscr{O}_X\big)$ [1, II (4.2), (4.3)].

Moreover, assume that $k = \mathbb{F}_q$ and that X is (a complete intersection of degree $d \geq 2$ and) proper and smooth over k. Then, recall that the Witt cohomology is, after the base extension to $K := \mathrm{Frac}(W(k))$, isomorphic to the maximal subspace $H^d_{\mathrm{cris}}\big(X/W(k)\big)_K^{<1}$ of the crystalline cohomology $H^d_{\mathrm{cris}}\big(X/W(k)\big)_K$ on which the Frobenius acts with slope < 1. If the formal group $H^d\big(X, \widehat{\mathbb{G}_{\mathrm{m}}}\big)$ is one-dimensional and if the height h of $H^d\big(X, \widehat{\mathbb{G}_{\mathrm{m}}}\big)$ mod p is finite, then, h is equal to the dimension of $H^d_{\mathrm{cris}}\big(X/W(k)\big)_K^{<1}$. In fact, both equal the rank of Cartier–Dieudonné module associated with the formal group [14, II, Remarques 2.15 (a)], [21, (A.13)]. Because the slope of Frobenius equals $(h-1)/h$, the first slope of the Newton polygon of the crystalline cohomology is zero if and only if $h = 1$ [5]. In this case, the unique eigenvalue of the Frobenius on $H^d_{\mathrm{cris}}\big(X/W(k)\big)_K$ that is a p-adic unit is called the unit-root of X_λ.

Theorem 2.7. *Let R be a flat $W(\mathbb{F}_q)$-algebra, and let Λ be an element of R. For each element c of $\mathbb{F}_q$, we denote the Teichmüller lift of c by $\widetilde{c}$. Let $\widetilde{X_\Lambda}$ be the hypersurface in $\mathbb{P}^n_R$ defined by*

$$\widetilde{F_\Lambda}(T) := \widetilde{c_1}T^{a_1} + \cdots + \widetilde{c_{n+1}}T^{a_{n+1}} - \Lambda T_1 \ldots T_{n+1} \in R[T_1, \ldots, T_{n+1}],$$

and assume that $\widetilde{X_\Lambda}$ is flat over R. Then, the Artin–Mazur functor $H^{n-1}(\widetilde{X_\Lambda}, \widehat{\mathbb{G}_{\mathrm{m}}})$ is a formal group whose logarithm $l(\tau)$ is given by

$$\sum_{m=0}^{\infty} (-\Lambda)^m {}_{\alpha}F_{\alpha-1}\left(\begin{matrix} \frac{-m}{\alpha}, \frac{-m+1}{\alpha}, \ldots, \frac{-m+\alpha-1}{\alpha} \\ \frac{1}{\alpha_1}, \ldots, \frac{\alpha_1-1}{\alpha_1}, 1, \frac{1}{\alpha_2}, \ldots, \frac{\alpha_2-1}{\alpha_2}, 1, \ldots, \frac{1}{\alpha_{n+1}}, \ldots, \frac{\alpha_{n+1}-1}{\alpha_{n+1}} \end{matrix}; \widetilde{C}\Lambda^{-\alpha}\right) \frac{\tau^{m+1}}{m+1}.$$

Here, C is the constant given in Definition 2.4 (and $\tilde{C}$ is the Teichmüller lift of C).

Proof. We already know that $H^{n-1}(\widetilde{X_\Lambda}, \widehat{\mathbb{G}_{\mathrm{m}}})$ is a formal group. A theorem of Stienstra [20, Theorem 1] shows that the coefficient of $\tau^{m+1}/(m+1)$ of its logarithm equals the coefficient of $T_1^m \ldots T_{n+1}^m$ of $\widetilde{F_\Lambda}^m$.

Now, let us calculate the coefficient of the latter. Looking at the binomial expansion

$$\widetilde{F_\Lambda}(T)^m = \sum_{m_1+\cdots+m_{n+1}+m'=m} \frac{m!}{m_1! \ldots m_{n+1}! m'!} \prod_{i=1}^{n+1} \widetilde{c_i}^{m_i} T^{m_i a_i} (-\Lambda T_1 \ldots T_{n+1})^{m'},$$

we notice that the coefficient of $T_1^m \dots T_{n+1}^m$ is the sum

$$\sum_{(m_1,\dots,m_{n+1},m')} \frac{m!}{m_1!\dots m_{n+1}!m'!} \cdot \widetilde{c_1}^{m_1} \dots \widetilde{c_{n+1}}^{m_{n+1}} (-\Lambda)^{m'},$$

where the index runs the vector $(m_1,\dots,m_{n+1},m')$ such that

$$\begin{aligned} m_1 a_{i1} + \dots + m_{n+1} a_{i,n+1} + m' &= m \quad (i = 1,\dots,n+1), \\ m_1 + \dots + m_{n+1} + m' &= m. \end{aligned}$$

By Proposition 2.2, an $(n+2)$-tuple $(m_1,\dots,m_{n+1},m)$ satisfies this condition if and only if the equation

$$(m_1,\dots,m_{n+1}) = k(\alpha_1,\dots,\alpha_{n+1}), \quad m' = m - k\alpha$$

holds for a natural number k. This shows that the coefficient equals

$$\sum_{k\geq 0, m\geq k\alpha} \frac{m!}{(k\alpha_1)!\dots(k\alpha_{n+1})!(m-k\alpha)!} \widetilde{c_1}^{k\alpha_1} \dots \widetilde{c_{n+1}}^{k\alpha_{n+1}} (-\Lambda)^{m-k\alpha}. \tag{3}$$

On the other hand, the number

$$(-\Lambda)^m {}_\alpha F_{\alpha-1}\left(\begin{matrix} \frac{-m}{\alpha}, \frac{-m+1}{\alpha}, \dots, \frac{-m+\alpha-1}{\alpha} \\ \frac{1}{\alpha_1},\dots,\frac{\alpha_1-1}{\alpha_1}, 1, \frac{1}{\alpha_2},\dots,\frac{\alpha_2-1}{\alpha_2}, 1,\dots,\frac{1}{\alpha_{n+1}},\dots,\frac{\alpha_{n+1}-1}{\alpha_{n+1}} \end{matrix} ; \widetilde{C}\Lambda^{-\alpha} \right). \tag{4}$$

equals, by definition of hypergeometric function,

$$\begin{aligned} & (-\Lambda)^m \sum_{k=0}^{\infty} \frac{\left(\frac{-m}{\alpha}\right)_k \left(\frac{-m+1}{\alpha}\right)_k \cdots \left(\frac{-m+\alpha-1}{\alpha}\right)_k}{\left(\frac{1}{\alpha_1}\right)_k \cdots \left(\frac{\alpha_1}{\alpha_1}\right)_k \cdots \left(\frac{1}{\alpha_{n+1}}\right)_k \cdots \left(\frac{\alpha_{n+1}}{\alpha_{n+1}}\right)_k} \left(\frac{\alpha^\alpha}{\Lambda^\alpha} \frac{\widetilde{c_1}^{\alpha_1}}{\alpha_1^{\alpha_1}} \cdots \frac{\widetilde{c_{n+1}}^{\alpha_{n+1}}}{\alpha_{n+1}^{\alpha_{n+1}}} \right)^k \\ = & \sum_{k=0}^{\infty} \frac{\alpha^{k\alpha} \left(\frac{-m}{\alpha}\right)_k \left(\frac{-m+1}{\alpha}\right)_k \cdots \left(\frac{-m+k\alpha-1}{\alpha}\right)_k}{A_{1,k} \dots A_{n+1,k}} \widetilde{C'_k} (-\Lambda)^{m-k\alpha}, \end{aligned}$$

where

$$A_{i,k} = \alpha_i^{k\alpha_i} \left(\tfrac{1}{\alpha_i}\right)_k \cdots \left(\tfrac{\alpha_i-1}{\alpha_i}\right)_k \left(\tfrac{\alpha_i}{\alpha_i}\right)_k \quad (i = 1,\dots,n+1)$$

and $C'_k = c_1^{k\alpha_1} \dots c_{n+1}^{k\alpha_{n+1}}$. Moreover, each summand is 0 if $k - m/\alpha \geq 0$. We, in fact, have $A_{i,k} = (k\alpha_i)!$ and

$$\begin{aligned} & (-1)^{k\alpha} \alpha^{k\alpha} \left(\tfrac{-m}{\alpha}\right)_k \left(\tfrac{-m+1}{\alpha}\right)_k \cdots \left(\tfrac{-m+k\alpha-1}{\alpha}\right)_k \\ = & (-1)^{k\alpha} (-m)(-m+1)\dots(-m+k\alpha-1) = \frac{m!}{(m-k\alpha)!}. \end{aligned}$$

This shows that two numbers (3) and (4) coincide with each other. □

2.3. **The unit-root.** Let $\mathscr{F}(x)$ denote the formal power series

$${}_\alpha F_{\alpha-1}\left(\begin{matrix} \frac{1}{\alpha}, \frac{2}{\alpha}, \dots, \frac{\alpha-1}{\alpha}, 1 \\ \frac{1}{\alpha_1},\dots,\frac{\alpha_1-1}{\alpha_1}, 1, \frac{1}{\alpha_2},\dots,\frac{\alpha_2-1}{\alpha_2}, 1,\dots,\frac{1}{\alpha_{n+1}},\dots,\frac{\alpha_{n+1}-1}{\alpha_{n+1}} \end{matrix} ; \widetilde{C}x \right) \in W(\mathbb{F}_q)[[x]]. \tag{5}$$

In this subsection, by a method of Stienstra and Beukers [21], we prove that $\mathscr{F}(x)$ gives us the information of the unit-root of X_λ. (Our argument is a generalization of an argument by Yu [22, Section 5].)

For positive integers m and s, let $\mathscr{F}_{m,s}(x)$ denote the polynomial obtained by truncating $\mathscr{F}(x)$ up to degree $mp^s - 1$. Let $\mathscr{R}$ denote the p-adic completion of the ring

$$W(\mathbb{F}_q)\left[x, \left(x\mathscr{F}_{1,1}(x)\right)^{-1}\right],$$

to which the Frobenius endomorphism σ on $W(\mathbb{F}_q)$ extends by $\sigma(x) = x^p$.

Lemma 2.8. *The formal power series*

$$f(x) := \frac{\mathscr{F}(x)}{\sigma\big(\mathscr{F}(x)\big)} \in W\big(\mathbb{F}_q\big)[[x]]$$

is actually an element of $\mathscr{R}$.

Proof. Define a polynomial $G_{\mu,s}(x)$, for positive integers μ and s, by

$$G_{\mu,s}(x) = {}_{\alpha}F_{\alpha-1}\left(\begin{matrix} \frac{-\mu p^s+1}{\alpha}, \frac{-\mu p^s+2}{\alpha}, \dots, \frac{-\mu p^s+\alpha}{\alpha} \\ \frac{1}{\alpha_1}, \dots, \frac{\alpha_1-1}{\alpha_1}, 1, \frac{1}{\alpha_2}, \dots, \frac{\alpha_2-1}{\alpha_2}, 1, \dots, \frac{1}{\alpha_{n+1}}, \dots, \frac{\alpha_{n+1}-1}{\alpha_{n+1}} \end{matrix}; \widetilde{C}x\right).$$

Let $G'_{\mu,s}(t)$ be the polynomial defined by

$$G'_{\mu,s}(t) = (-t)^{\mu p^s-1} G_{\mu,s}(t^{-\alpha}).$$

This is the coefficient of $\tau^{\mu p^s}/\mu p^s$ in the logarithm $l(\tau)$ in Theorem 2.7, applied with R being the p-adic completion $\mathscr{S}$ of the ring

$$W\big(\mathbb{F}_q\big)\left[t, \big(t\mathscr{F}_{1,1}(t^{-\alpha})\big)^{-1}\right]$$

and with $\Lambda = t$. Now, since we have $G'_{1,1}(t) \equiv t^{p-1}\mathscr{F}_{1,1}(t^{-\alpha}) \pmod{p}$ by a straightforward observation, a general fact on formal group [21, (A.8), (i)$\Longrightarrow$(v)] shows that there exists an element g' of $\mathscr{S}$ independent of μ and s that satisfies

$$G'_{\mu,s+1}(t) \equiv g'(t) \cdot \sigma\big(G'_{\mu,s}(t)\big) \pmod{p^{s+1}} \qquad (\mu, s \geq 1).$$

Thus, we have

$$t^{p-1} G_{\mu,s+1}(t^{-\alpha}) \equiv g'(t)\sigma\big(G_{\mu,s}(t^{-\alpha})\big) \pmod{p^{s+1}}.$$

Since the polynomial $G_{\mu,s}(x)$ converges p-adically to $\mathscr{F}(x)$ as $s \to \infty$, the power series $f(x)$ equals $t^{-(p-1)}g'(t)$, with $x = t^{-\alpha}$. This element, therefore, lies in the intersection of $W\big(\mathbb{F}_q\big)[[x]]$ and the ring $\mathscr{S}$, which equals $\mathscr{R}$. □

Theorem 2.9. *Let* λ *be an element of* $\mathbb{F}_q^\times$ *such that* X_λ *is smooth, and let* $\widetilde{\lambda}$ *be the Teichmüller lift of* λ. *Then, the first slope of the Newton polygon of* $H^{n-1}_{\mathrm{cris}}\big(X_\lambda/W(\mathbb{F}_q)\big)$ *is zero if and only if* $\mathscr{F}_{1,1}(\lambda^{-\alpha}) \neq 0$ *in* $\mathbb{F}_q$. *In this case, the unit-root of* X_λ *equals*

$$\prod_{i=0}^{r-1} \sigma^i\big(f(\widetilde{\lambda}^{-\alpha})\big), \tag{6}$$

where $q = p^r$.

Proof. Put $a := \widetilde{\lambda}^{p-1} f(\widetilde{\lambda}^{-\alpha})$ where f is the element in Lemma 2.8 and put, for each $s \geq 0$, $a_s := a\sigma(a)\dots\sigma^s(a)$. We denote by G' the formal group law over $W\big(\mathbb{F}_q\big)$ with logarithm $l'(\tau) = \sum_{s=0}^{\infty} a_s \tau^{p^s}/p^s$, and denote by $\widetilde{X_\lambda}$ the variety $\widetilde{X_\Lambda}$ with $\Lambda = \widetilde{\lambda}$ in the notation of Theorem 2.7. Then, G' is strictly isomorphic to the formal group law realizing $H^{n-1}\big(\widetilde{X_\lambda}, \widehat{\mathbb{G}_{\mathrm{m}}}\big)$ by [21, (A.9), (iii)$\Longrightarrow$(i)] because a satisfies the condition (iii) there. The first half of the theorem follows from Lemma 2.6 because $a \equiv \mathscr{F}_{1,1}(\lambda^{-\alpha}) \pmod{p}$. The proof of the second half goes exactly as in [22, p.76, proof of Theorem 4.3 (2)]. □

3. FACTORIZATION OF THE ZETA FUNCTION.

3.1. **Preparation for stating the main theorem.** In this subsection, we introduce assumptions and notations which we need to state the main theorem of this article, that is, the precise description of the zeta function of X_λ using hypergeometric functions over finite fields.

Recall from Subsection 2.1 the definition of positive integers $\alpha_1,\dots,\alpha_{n+1}$ and α; the vector ${}^{\mathrm{t}}(\alpha_1,\dots,\alpha_{n+1})$ is the kernel of the homomorphism $\mathbb{Z}^{n+1}\to\mathbb{Z}^{n+1}$ obtained by the matrix $A'=\left(a_{ij}-1\right)_{i,j}$, and $\alpha=\sum_{i=1}^{n+1}\alpha_i$. Let N be a positive integer divisible by all α_i's and α, and let

$$f_N\colon \frac{\left(\mathbb{Z}/N\mathbb{Z}\right)^{n+1}}{{}^{\mathrm{t}}(\alpha_1,\dots,\alpha_{n+1})}\to\left(\mathbb{Z}/N\mathbb{Z}\right)^{n+1}$$

denote the morphism defined by the endomorphism of $\left(\mathbb{Z}/N\mathbb{Z}\right)^{n+1}$ defined by the matrix A' mod N.

Let $d_1,\dots,d_n$ be non-zero elementary divisors of A', and put $d:=d_1\dots d_n$. Then, $\mathrm{Ker}\left(f_N\right)$ consists of d elements by the assumption on N because $\Im\left(f_N\right)\cong\oplus_{i=1}^{n}d_i\mathbb{Z}/N\mathbb{Z}$. In the rest of this section, we assume that $q-1$ is divisible by all α_i's and by α, and we fix d vectors $s_0=(0,\dots,0),s_1,\dots,s_{d-1}\in\{0,\dots,q-2\}^{n+1}$ that represent $\mathrm{Ker}\left(f_{q-1}\right)$. Let us write $s_j={}^{\mathrm{t}}(s_{1j},\dots,s_{n+1,j})$.

Assumption 3.1. *(i) $q-1$ is divisible by all α_i's and by α.*
(ii) For all i and j, each s_{ij} is divisible by α_i and $|s_j|$ is divisible by α.

Note that, provided that $\mathbb{F}_q$ satisfies Assumption 2.3, then a finite extension of $\mathbb{F}_q$ also satisfies Assumption 3.1. In fact, it is clear that $\mathbb{F}_q$ satisfies Assmption 3.1 (i) after a finite extension. Then, replacing q by q^r invokes the multiplication by $(q^r-1)/(q-1)$, which is an isomorphism $\mathrm{Ker}\left(f_{q-1}\right)\to\mathrm{Ker}\left(f_{q^r-1}\right)$ (it is an isomorphism because it is injective and because the order of the target equals that of the source).

Under Assumption 3.1, we put $t_{ij}:=s_{ij}/\alpha_i$ and $t_j:=|s_j|/\alpha$. Since $|s_j|<n(q-1)<\alpha(q-1)$, we have $t_j\in\{0,1,\dots,q-2\}$, and $t_j=0$ if and only if $j=0$.

Assumption 3.2. *Let $J=\{j_1,\dots,j_t\}$ be an arbitrary subset $\{1,2,\dots,n+1\}$ with $t\geq(n+1)/2$ elements. Let $\sigma(J)$ denote the number of indices $i\in\{1,\dots,n+1\}$ that satisfies $a_{ij}=0$ for all $j\notin J$, and let $i_1,\dots,i_{\sigma(J)}$ be all such indices (they are elements of J). Under this notation, we assume that all elementary divisors of the matrix*

$$\begin{pmatrix} a_{j_1,i_1} & \dots & a_{j_1,i_{\sigma(J)}}\\ & \vdots & \\ a_{j_t,i_1} & \dots & a_{j_t,i_{\sigma(J)}}\\ 1 & \dots & 1\end{pmatrix}\in M_{t+1,\sigma(J)}\left(\mathbb{Z}\right)$$

divide $q-1$.

Here, note that this matrix is of rank $\sigma(J)$ and therefore all elementary divisors are non-zero. In fact, since $\{i_1,\dots,i_{\sigma(J)}\}$ is a subset of J, the matrix above contains

$$\begin{pmatrix} a_{i_1,i_1} & \dots & a_{i_1,i_{\sigma(J)}}\\ & \vdots & \\ a_{i_{\sigma(J)},i_1} & \dots & a_{i_{\sigma(J)},i_{\sigma(J)}}\\ 1 & \dots & 1\end{pmatrix}$$

as a minor matrix. Then, for each element of the kernel of the homomorphism defined by this matrix, all the coefficients have the same sign as in the proof of Proposition 2.2. This forces the element to be zero since all the entries of the lowest row are 1.

Example 3.3. *(i) Consider the Dwork family* $F_0(T) = T_1^{n+1} + \cdots + T_{n+1}^{n+1}$*; recall that* $\alpha = {}^{\mathrm{t}}(1,1,\dots,1)$*. Then, Assumption 3.1 (i) is equivalent to* $q \equiv 1 \pmod{n+1}$*, and in this case Assumption 3.1 (ii) and Assumption 3.2 are automatically satisfied. In fact, first, the elementary divisors of* A' *are* $1, n+1, \dots, n+1, 0$*. Since* $\mathrm{Ker}\left(f_{q-1}\right)$ *is generated by n vectors* ${}^{\mathrm{t}}\big((q-1)/(n+1), n(q-1)/(n+1), 0, \dots, 0\big)$, ${}^{\mathrm{t}}\big(0, (q-1)/(n+1), n(q-1)/(n+1), 0, \dots, 0\big), \dots, {}^{\mathrm{t}}\big(0, \dots, 0, (q-1)/(n+1), n(q-1)/(n+1)\big)$ *modulo* $q-1$*, there are no extra conditions concerning this vector. Moreover, all elementary divisors of the matrices in Assumption 3.2 above divide* $n+1$.

(ii) Consider the example from Example 2.5 (ii), that is, $F_0(T) = T_1^4 + T_1T_2^3 + T_3^4 + T_4^4$*; recall that* $\alpha = {}^{\mathrm{t}}(2,4,3,3)$*. Then, Assumption 3.1 (i) is equivalent to* $q \equiv 1 \pmod{12}$*. Moreover, Assumption 3.1 (ii) is equivalent to* $q \equiv 1 \pmod{24}$*, and in this case Assumption 3.2 is automatically satisfied. In fact, first, the elementary divisors of* A' *is* $1,1,4,0$*. Since* $\mathrm{Ker}\left(f_{q-1}\right)$ *is generated by the vector* ${}^{\mathrm{t}}\big((q-1)/4, 0, 3(q-1)/4, 0\big)$*, we have to impose the condition "modulo 24", not just "modulo 12". The J's to be taken care of is* $\{2,3,4\}, \{1,3,4\}, \{1,2,3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{3,4\}$ *and these sets with* 3 *and* 4 *reversed. In fact, all elementary divisors to be considered divide 4; for example, the matrices in Assumption 3.2 corresponding to* $J = \{2,3,4\}, \{1,3,4\}, \{3,4\}$ *are*

$$\begin{pmatrix} 0 & 0 \\ 4 & 0 \\ 0 & 4 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \\ 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 4 & 0 \\ 0 & 4 \\ 1 & 1 \end{pmatrix}$$

respectively.

Notation 3.4. *(i) In the rest of this article, we fix a generator* ρ *of the character group* $\widehat{\mathbb{F}_q^{\times}}$.

(ii) For each natural number β *dividing* $q-1$*, we put* $\varphi_\beta := \rho^{(q-1)/\beta}$.

Notation 3.5. *(i) For each* $j = 0, \dots, d-1$*, we define the function* $r \mapsto F(j)_r$ *on* $\mathbb{Z}_{>0}$ *as follows. If* $j = 0$*, then* $F(0)_r$ *equals*

$$_\bullet\widetilde{F}_\bullet\mathrm{Red}\left(\begin{matrix} [\varphi_\alpha] \\ [\varphi_{\alpha_1}], \dots, [\varphi_{\alpha_{n+1}}] \end{matrix}; C\lambda^{-\alpha}\right)_{\mathbb{F}_{q^r}},$$

where $[\varphi_\beta]$ *denotes the sequence* $\varepsilon, \varphi_\beta^1, \dots, \varphi_\beta^{\beta-1}$*. If* $j > 0$*, then* $F(j)_r$ *equals*

$$q^{\delta_{|s_j|}-1}{}_\bullet\widetilde{F}_\bullet\mathrm{Red}\left(\begin{matrix} \rho^{t_j}[\varphi_\alpha] \\ \rho^{t_{1j}}[\varphi_{\alpha_1}], \dots, \rho^{t_{n+1,j}}[\varphi_{\alpha_{n+1}}] \end{matrix}; C\lambda^{-\alpha}\right)_{\mathbb{F}_{q^r}},$$

where $\psi[\varphi_\beta]$ *denotes the sequence* $\psi, \psi\varphi_\beta^1, \dots, \psi\varphi_\beta^{\beta-1}$*, and where*

$$\delta_{|s_j|} := \begin{cases} 1 & \textit{if } |s_j| \equiv 0 \pmod{q-1}, \\ 0 & \textit{if } |s_j| \not\equiv 0 \pmod{q-1}. \end{cases}$$

(ii) For each $j = 0, \dots, d-1$, and $r \ge 1$, we put

$$\gamma(j)_r := \prod_{i=1}^{n+1} \rho_r^{s_{ij}}\big(\alpha_i^{-1} c_i\big) \cdot \rho_r^{s_j}\big((-\lambda)^{-1}\alpha\big) \times \prod_{i=1}^{n+1} \left\{ G\big(\rho_r^{-t_{ij}}\big) \prod_{b_i=1}^{\alpha_i-1} \frac{G\big(\rho_r^{-t_{ij}}\varphi_{\alpha_i,r}^{b_i}\big)}{G\big(\varphi_{\alpha_i,r}^{b_i}\big)} \right\} G\big(\rho_r^{t_j}\big) \prod_{b=1}^{\alpha-1} \frac{G\big(\rho_r^{t_j}\varphi_{\alpha,r}^{b}\big)}{G\big(\varphi_{\alpha,r}^{b}\big)}.$$

For a positive integer r, the element $\gamma(j)_r$ is similarly defined by replacing ρ by ρ_r and φ_β's by $\varphi_{\beta,r}$'s for $\beta \in \{\alpha_1, \dots, \alpha_{n+1}, \alpha\}$.

(iii) For $r \ge 1$, we put

$$u_r := \sum_{\substack{J \subset \{1,\dots,n+1\} \\ \#J \ge (n+1)/2}} \sum_{i=0}^{\#J-\sigma(J)} (-1)^{\#J-\sigma(J)-i} q^{r(i-1)} \sum \prod_{j=1}^{\sigma(J)} G\big(\chi_{j,r}^{-1}\big)\chi_{j,r}(c_j);$$

here, in the most inner sum, the index runs through all elements $(\chi_1, \dots, \chi_{\sigma(J)})$ in $\mathrm{Ker}\big(\varphi(\widetilde{A})\big)$ such that exactly $n-2i+1$ components are non-trivial.

Lemma 3.6. (i) *The functions $r \mapsto \gamma(j)_r$ for each $j = 0, \dots, d-1$ are pure q-Weil functions. If $j = 0$, then $\gamma(0)_r = 1$ for all $r > 0$. If $j \ne 0$, then the weight of $\gamma_r(j)$ is $\#\big\{ i \in \{1,2,\dots,n+1\} \bigm| s_{ij} \ne 0 \big\} + 1 - \delta_{|s_j|}$.*

(ii) *The functions $r \mapsto \gamma(j)_r F(j)_r$ for each $j = 0, \dots, d-1$ are pure q-Weil functions of weight $n-1$.*

(iii) *$r \mapsto u_r$ is a q-Weil functions of weight $n-1$.*

Proof. (i) By the definition of t_{ij}'s, $\rho^{-t_{ij}}\varphi_{\alpha_i}^{b_i}$ $(b_i = 1, \dots, \alpha_i - 1)$ cannot be the trivial character unless $t_{ij} \ne 0$. If no $\rho^{t_j}\varphi_\alpha^b$'s are trivial, that is, if $|s_j| \not\equiv 0 \pmod{q-1}$, then the weight is $\#\big\{ i \in \{1,2,\dots,n+1\} \bigm| s_{ij} \ne 0 \big\} + 1$ by Proposition 1.9. If $|s_j| \equiv 0 \pmod{q-1}$, then exactly one $G\big(\rho^{t_j}\varphi_\alpha^b\big)$ is of weight 0, and the claim follows.

(ii) If $j = 0$, then the function $r \mapsto F(0)_r$ gives a pure q-Weil function of weight $n-1$ by Proposition 1.10 and $\gamma(0)_r = 1$, which shows the claim. Assume that $j \ne 0$. Then, the function $r \mapsto F(j)_r$ gives a pure q-Weil function of weight $\#\big\{ i \in \{1,\dots,n+1\} \bigm| s_{ij} = 0 \big\} - 1 - \delta_{|s_j|} + 2(-1 + \delta_{|s_j|})$ by Proposition 1.10. Therefore, we see that the function $r \mapsto \gamma(j)_r F(j)_r$ is, together with (i), a pure q-Weil function of weight $n-1$.

(iii) is a direct consequence of Proposition 1.9. □

3.2. **Statement of the main theorem and the strategy of our proof.** Recall from Definition 1.5 that, for a function $\mathbb{Z}_{>0} \to \overline{\mathbb{Q}}$, we introduced the associated zeta function. Here we consider the q-Weil functions $r \mapsto u_r$ and $r \mapsto \gamma(j)_r F(j)_r$, and denote their associated zeta functions by $\zeta\big(u_r\big)$ and $\zeta\big(\gamma(j)_r F(j)_r\big)$ respectively. Now, we may state the main theorem.

Theorem 3.7. *Under Assumptions 3.1 and 3.2, let λ be an element of $\mathbb{F}_q^\times$ such that X_λ is smooth and $C \ne \lambda^\alpha$. Define the polynomial $P(T) \in \mathbb{Z}[T]$ by*

$$\zeta\big(X_\lambda, T\big) = \frac{P(T)^{(-1)^n}}{(1-T)(1-qT)\dots(1-q^{n-1}T)}.$$

Then, $P(T)$ equals

$$\zeta\big(u_r\big)(T)\big(1 - q^{(n-1)/2}T\big)^D \prod_{i=0}^{d-1} \zeta\big(\gamma(j)_r F(j)_r\big)(T),$$

where the number D is defined to be the number of subsets $J \subset \{1, 2, \dots, n+1\}$ such that $\#J = (n+1)/2$ and that for all $i = 1, \dots, n+1$ there exists $j \notin I$ such that $a_{ij} \geq 1$ (in particular, it is zero if n is even).

Remark 3.8. *(i) Note that the parameters of the hypergeometric function $F(0)_r$ corresponds to the parameters of $\mathscr{F}(x)$ in the p-adic situation (5).*

(ii) Lemma 3.6 (ii) shows that each $\zeta\big(\gamma(j)_r F(j)_r\big)(T)$ is a polynomial. Moreover, this is the Frobenius trace of an irreducible *$\overline{\mathbb{Q}_\ell}$-sheaf on $\mathbb{G}_{\mathrm{m}}$ at a point* [15, Theorem 8.4.2]*, as we explained in Subsection 1.1.*

(iii) $\zeta(u_r)(T)$ is a rational function all of whose reciprocal zeros and poles belong to $\mathbb{Q}(\mu_{q-1})$, where μ_{q-1} is the subgroup of $\overline{\mathbb{Q}}^\times$ of order $q-1$.

In order to prove this theorem, it clearly suffices to show the following proposition.

Proposition 3.9. *Under Assumptions 3.1 and 3.2, the number of $\mathbb{F}_{q^r}$-rational points of X_λ equals*

$$\sum_{i=0}^{n-1} (q^r)^i + u_r + D(q^r)^{(n-1)/2} + (-1)^n \sum_{j=0}^{d-1} \gamma(j)_r F(j)_r,$$

where $F(j)_r$ is the function defined in the statement of the theorem.

The key point of our proof of this proposition is to calculate the number of $\mathbb{F}_{q^r}$-rational points of X_λ "modulo q-Weil function of weight $\neq n-1$"; in fact, we know from the Weil conjecture that the "weight $\neq n-1$ part" of this sum equals $1 + q + \cdots + q^{n-1}$. In the next subsection, we prove the following assertion in the next subsection, from which Proposition 3.9 directly follows.

Proposition 3.10. *Under Assumptions 3.1 and 3.2, the function $r \mapsto X_\lambda\big(\mathbb{F}_{q^r}\big)$ is the sum of the function*

$$r \mapsto \sum_{i=0}^{n-1} (q^r)^i + u_r + D(q^r)^{(n-1)/2} + \sum_{j=0}^{d-1} \gamma(j)_r F(j)_r$$

and a q-Weil function of weight $\neq n-1$.

3.3. **Proof: Counting rational points.** In this subsection, we assume Assumptions 3.1 and 3.2, fix an element λ of $\mathbb{F}_q^\times$ such that X_λ is smooth and $C \neq \lambda^\alpha$, and prove Proposition 3.10.

First, we recall a classical formula that expresses the number of rational points of arbitrary hypersurface in $\mathbb{G}_m$ in terms of Gauss sums.

Notation 3.11. *Let n and N be positive integers and let $M = (m_{ij})_{i,j}$ be an $n \times N$ matrix with coefficients in $\mathbb{Z}$. We denote by $\varphi(M)$ the natural homomorphism $\big(\widehat{\mathbb{F}_q^\times}\big)^N \to \big(\widehat{\mathbb{F}_q^\times}\big)^n$ defined by M, which is explicitly expressed as*

$$\varphi(M)\big((\chi_i)_{i=1,\dots,N}\big) = \left(\chi_1^{m_{j1}} \dots \chi_n^{m_{jN}}\right)_{j=1,\dots,n}.$$

We always regard elements of $\big(\widehat{\mathbb{F}_q^\times}\big)^N$ and $\big(\widehat{\mathbb{F}_q^\times}\big)^n$ as column vectors.

The following general theorem is classical [4], [7]; the readers also can find the proof in this context in [19, Proposition 3].

Proposition 3.12. *Let n and N be positive integers, let $c_1, \dots, c_N$ be elements of $\mathbb{F}_q^\times$, and let $R = (r_{ij})_{i,j} \in M_{n,N}(\mathbb{Z})$ be an $n \times N$ matrix with coefficients in $\mathbb{Z}$. Define a polynomial*

$f(X_1,\dots,X_n)\in\mathbb{F}_q[X_1,\dots,X_n]$ by

$$f(X_1,\dots,X_n)=\sum_{j=1}^{N}c_jX_1^{r_{1j}}\dots X_n^{r_{nj}}.$$

Then, the number of n-tuples $(x_1,\dots,x_n)\in(\mathbb{F}_q^\times)^n$ satisfying $f(x_1,\dots,x_n)=0$ equals

$$\frac{(q-1)^n}{q}+\frac{(q-1)^{n+1-N}}{q}\sum_{(\chi_1,\dots,\chi_N)\in\operatorname{Ker}(\varphi(\widetilde{R}))}\prod_{j=1}^{N}G(\chi_j^{-1})\chi_j(c_j),$$

where $\widetilde{R}$ is the $(n+1)\times N$ matrix defined by

$$\widetilde{R}=\begin{pmatrix} & R & \\ \hline 1 & \dots & 1\end{pmatrix}.$$

Now, let $X_\lambda(\mathbb{F}_q)_0$ be the set of $\mathbb{F}_q$-rational points $[x_1:\dots:x_{n+1}]$ of X_λ at least one of whose coordinates is zero, and set $X_\lambda(\mathbb{F}_q)_*:=X_\lambda(\mathbb{F}_q)\setminus X_\lambda(\mathbb{F}_q)_0$.

Proposition 3.13. *The function $r\mapsto\#X_\lambda(\mathbb{F}_{q^r})_0$ is the sum of the function*

$$r\mapsto\sum_{i=0}^{n-1}(q^r)^i+D(q^r)^{(n-1)/2}-\frac{(q^r-1)^n}{q^r}+u_r$$

and a q-Weil function of weight $\neq n-1$.

Proof. For a proper non-empty subset $J\subset\{1,\dots,n+1\}$ and a positive integer r, denote by $s_r(J)$ the number of $\mathbb{F}_{q^r}$-rational points $[x_1:\dots:x_{n+1}]$ of X_λ such that $x_j\neq0$ if and only if $j\in J$. Then, we have

$$\#X_\lambda(\mathbb{F}_{q^r})_0=\sum_{t=1}^{n}\sum_{\#J=t}s_r(J).$$

By Lemma 3.14 below, each $s_r(J)$ is of the form

$$\frac{(q^r-1)^{t-1}}{q^r}+N'_J(q^r)^{(n-1)/2}+\sum_{i=0}^{\#J-\sigma(J)}(-1)^{\#J-\sigma(J)-i}q^{i-1}\sum\prod_{j=1}^{\sigma(J)}G(\chi_j^{-1})\chi_j(c_j)$$

plus a q-Weil function of weight $\neq n-1$, where N'_J is the number given in the statement of Lemma 3.14. This shows that $\#X_\lambda(\mathbb{F}_{q^r})_0$ is written as

$$\sum_{t=1}^{n}\binom{n+1}{t}\frac{(q^r-1)^{t-1}}{q^r}+D(q^r)^{(n+1)/2}+u_r \tag{7}$$

plus a q-Weil function of weight $\neq n-1$. Since

$$\sum_{t=0}^{n+1}\binom{n+1}{t}\frac{(q^r-1)^t}{q^r}=(q^r)^n,$$

the first term of (7) equals

$$\frac{(q^r)^n}{q^r-1}-\frac{1}{q^r(q^r-1)}-\frac{(q^r-1)^n}{q^r}=\frac{(q^r)^n-1}{q^r-1}-\frac{(q^r-1)^n}{q^r}+\frac{1}{q^r}.$$

□

Lemma 3.14. *Let J be a subset of $\{1,\dots,n+1\}$ that satisfies $1 \le \#J \le n$. We define a number N'_J as follows; it is 1 if $\#J = (n+1)/2$ and for each $i \in \{1,2,\dots,n+1\}$ there exists $j \notin J$ with $a_{ji} \ge 1$; it is 0 otherwise. Then, the function*

$$r \mapsto s_r(J) := \#\left\{ [x_1 : \cdots : x_{n+1}] \in X_\lambda\left(\mathbb{F}_{q^r}\right) \;\middle|\; x_j \neq 0 \text{ if and only if } j \in J \right\}.$$

can be written as the sum of the function

$$\frac{(q^r-1)^{\#J-1}}{q^r} + N'_J (q^r)^{(n-1)/2} + \sum_{i=0}^{\#J-\sigma(J)} (-1)^{\#J-\sigma(J)-i} q^{i-1} \sum \prod_{j=1}^{\sigma(J)} G\big(\chi_j^{-1}\big)\chi_j(c_j)$$

and a q-Weil function of weight $\neq n-1$. Here, in the most inner sum, the index runs through all elements $(\chi_1,\dots,\chi_{\sigma(J)})$ of $\operatorname{Ker}(\varphi(\widetilde{A}))$ such that exactly $n-2i+1$ components are non-trivial.

Proof. If $\#J \le n/2$, then $s_r(J)$ can be considered as the number of $\mathbb{F}_{q^r}$-rational points of a closed subscheme of $\mathbb{G}_{\mathrm{m},\mathbb{F}_q}^{\#J-1}$; since $\#J-1 \le n/2-1$, each term of the function in the statement is a q-Weil function of weight $\le n-2$. Therefore, we may and do assume that $\#J \ge (n+1)/2$. If $N'_J = 1$, then the claim is also trivial since X_J is isomorphic to $\mathbb{G}_{,\mathbb{F}_\mathrm{q}}^{(n+1)/2-1}$; now we assume that $N'_J = 0$. For simplicity, we change the coordinates if necessary, and assume that $J = \{1,\dots,\#J\}$ and that $\{i_1,\dots,i_{\sigma(J)}\} = \{1,\dots,\sigma(J)\}$ in the notation in Assumption.

Now, by Proposition 3.12,

$$\begin{aligned} s_1(J) &= \frac{1}{q-1}\#\left\{ (x_1,\dots,x_{\#J}) \in \mathbb{F}_q^\times \;\middle|\; c_1 x^{a_1} + \cdots + c_s x^{a_{\sigma(J)}} = 0 \right\} \\ &= \frac{(q-1)^{\#J-1}}{q} + \frac{(q-1)^{\#J-\sigma(J)}}{q} \sum_{(\chi_1,\dots,\chi_{\sigma(J)}) \in \operatorname{Ker}(\varphi(\widetilde{A_J}))} \prod_{j=1}^{\sigma(J)} G\big(\chi_j^{-1}\big)\chi_j(c_j), \end{aligned}$$

(8)

where A_J denotes the matrix

$$\begin{pmatrix} a_{11} & \dots & a_{1,\sigma(J)} \\ & \vdots & \\ a_{\#J,1} & \dots & a_{\#J,\sigma(J)} \end{pmatrix}.$$

Now, let us denote by $d'_1,\dots,d'_{\sigma(J)}$ the elementary divisors of the $(\#J+1)\times\sigma(J)$ matrix $\widetilde{A_J}$. Then, by hypothesis, we see as in the argument after Assumption 3.2 that the second term of (8) is $(q-1)$ times the sum of $d'_1 \dots d'_{\sigma(J)}$ numbers that are the products of $\sigma(J) \le \#J$ Gauss sums.

Since we may do this computation in the same way for various r, the statement similar to the argument after Assumption 3.2 and Proposition 1.9 shows the claim. □

Now, let us investigate the function $r \mapsto \#X_\lambda\left(\mathbb{F}_{q^r}\right)_*$.

Proposition 3.15. *The function $r \mapsto \#X_\lambda\left(\mathbb{F}_{q^r}\right)_*$ is the sum of the function*

$$r \mapsto \frac{(q^r-1)^n}{q^r} + (-1)^n \sum_{i=0}^{d-1} \gamma(j)_r F(j)_r$$

and a q-Weil function of weight $\le n-2$.

Proof. In this proof, we only investigate the number $X_\lambda(\mathbb{F}_{q^r})_*$ for $r=1$ because we may compute it for general r in exactly the same way.

By using Proposition 3.12, $\#X_\lambda(\mathbb{F}_q)_*$ equals

$$\frac{(q-1)^n}{q}+\frac{1}{q(q-1)}\sum_{{}^{\mathrm t}(\chi_1,\dots,\chi_{n+2})\in\operatorname{Ker}\varphi(\widetilde{A})}\prod_{i=1}^{n+1}G(\overline{\chi_i})\chi_i(c_i)\cdot G(\overline{\chi_{n+2}})\chi_{n+2}(-\lambda). \tag{9}$$

Let us describe the kernel of $\varphi(\widetilde{A})$. Recall that we fixed a generator ρ of $\widehat{\mathbb{F}_q^\times}$ in Notation 3.4, and take $k_i\in\mathbb{Z}$ such that $\chi_i=\rho^{k_i}$ for $i=1,\dots,n+2$. With this notation, the definition of s_{ij}'s shows that ${}^{\mathrm t}(\chi_1,\dots,\chi_{n+2})$ is an element of $\operatorname{Ker}(\varphi(\widetilde{A}))$ if and only if there exist an index $j=0,\dots,d-1$ and a number $a=0,\dots,q-2$ such that

$$k_i\equiv s_{ij}+a\alpha_i\quad(i=1,\dots,n+1)\quad\text{and}\quad k_{n+2}\equiv-|s_j|-a\alpha;$$

the choice of j and a is unique. Now, we have shown that the second term of (9) is

$$\begin{aligned}&\sum_{j=0}^{d-1}\sum_{a=0}^{q-2}\left\{\prod_{i=1}^{n+1}G(\rho^{-s_{ij}-a\alpha_i})\cdot G(\rho^{|s_j|+a\alpha})\prod_{i=1}^{n+1}\rho^{s_{ij}+a\alpha_i}(c_i)\rho^{-|s_j|-a\alpha}(-\lambda)\right\}\\ =\;&\sum_{j=0}^{d-1}\left\{\prod_{i=1}^{n+1}\rho^{s_{ij}}(c_i)\cdot\rho^{-|s_j|}(-\lambda)\sum_{a=0}^{q-2}\prod_{i=1}^{n+1}G(\rho^{-s_{ij}-a\alpha_i})\cdot G(\rho^{|s_j|+a\alpha})\rho^a\left((-1)^\alpha\prod_{i=1}^{n+1}c_i^{\alpha_i}\cdot\lambda^{-\alpha}\right)\right\}.\end{aligned}$$

For each j, we have $s_{ij}=\alpha_i t_{ij}$ and $|s_j|=\alpha t_j$, therefore Lemma 3.16 below shows that

$$\begin{aligned}&\frac{1}{q-1}\sum_{a=0}^{q-2}\prod_{i=1}^{n+1}G(\rho^{-s_{ij}-a\alpha_i})\cdot G(\rho^{|s_j|+a\alpha})\rho^a\left((-1)^\alpha\prod_{i=1}^{n+1}c_i^{\alpha_i}\cdot\lambda^{-\alpha}\right)\\ =\;&(-1)^n\prod_{i=1}^{n+1}\rho^{t_{ij}}(\alpha_i^{-\alpha_i})\cdot\rho^{t_j}(\alpha^\alpha)\prod_{i=1}^{n+1}\left\{G(\rho^{-t_{ij}})\prod_{b_i=1}^{\alpha_i-1}\frac{G(\rho^{-t_{ij}}\varphi_{\alpha_i}^{b_i})}{G(\varphi_{\alpha_i}^{b_i})}\right\}G(\rho^{t_j})\prod_{b=1}^{\alpha-1}\frac{G(\rho^{t_j}\varphi_\alpha^b)}{G(\varphi_\alpha^b)}\\ &\times{}_\alpha\widetilde{F}_\alpha\left(\begin{matrix}\rho^{t_j}[\varphi_\alpha]\\ \rho^{t_{1j}}[\varphi_{\alpha_1}],\dots,\rho^{t_{n+1,j}}[\varphi_{\alpha_{n+1}}]\end{matrix};C\lambda^{-\alpha}\right)_{\mathbb{F}_q}.\end{aligned}$$

Finally, the second term of (9) for various r is the sum of the function $r\mapsto(-1)^n\sum_{j=0}^{d-1}\gamma(j)_rF(j)_r$ and a q-Weil function of weight $\le n-2$ by Proposition 1.12; use (i) for $j=0$ and for $j\neq0$ such that $\delta_{|s_j|}=1$, and use (ii) for the other j's. □

Lemma 3.16. *Let $\alpha_1,\dots,\alpha_{n+1}$ be positive integers, put $\alpha:=\alpha_1+\cdots+\alpha_{n+1}$, and assume that q is congruent to 1 modulo all α_i's and modulo α.*

Let $A_1,\dots,A_{n+1},B$ be characters on $\mathbb{F}_q^\times$. Then, we have the equation

$$\begin{aligned}&\sum_{\chi\in\widehat{\mathbb{F}_q^\times}}\prod_{i=1}^{n+1}G((\overline{A_i\chi})^{\alpha_i})G((B\chi)^\alpha)\chi((-1)^\alpha x)\\ =\;&(-1)^n(q-1)\prod_{i=1}^{n+1}A_i(\alpha_i^{-\alpha_i})\cdot B(\alpha^\alpha)\prod_{i=1}^{n+1}\left\{G(\overline{A_i})\prod_{b_i=1}^{\alpha_i-1}\frac{G(\overline{A_i}\varphi_{\alpha_i}^{b_i})}{G(\varphi_{\alpha_i}^{s_i})}\right\}\cdot G(B)\prod_{b=1}^{\alpha-1}\frac{G(B\varphi_\alpha^b)}{G(\varphi_\alpha^b)}\\ &\times{}_\alpha\widetilde{F}_\alpha\left(\begin{matrix}B[\varphi_\alpha]\\ A_1[\varphi_{\alpha_1}],\dots,A_{n+1}[\varphi_{\alpha_{n+1}}]\end{matrix};\frac{\alpha^\alpha}{\alpha_1^{\alpha_1}\dots\alpha_{n+1}^{\alpha_{n+1}}}x\right)_{\mathbb{F}_q}.\end{aligned}$$

Proof. Davenport–Hasse relation [3, (0.9_1)] shows that

$$G\big((\overline{A_i\chi})^{\alpha_i}\big) = -G\big(\overline{A_i\chi}\big)\prod_{s_i=1}^{\alpha_i-1}\frac{G\big(\overline{A_i\chi}\varphi_{\alpha_i}^{s_i}\big)}{G\big(\varphi_{\alpha_i}^{s_i}\big)}\big(A_i\chi\big)\big(\alpha_i^{-\alpha_i}\big),$$

$$G\big((B\chi)^{\alpha}\big) = -G\big(B\chi\big)\prod_{s=1}^{\alpha-1}\frac{G\big(B\chi\varphi_{\alpha}^{s}\big)}{G\big(\varphi_{\alpha}^{s}\big)}\big(B\chi\big)\big(\alpha^{\alpha}\big).$$

Therefore, the left-hand side of the equation in the statement equals

$$(-1)^n\prod_{i=1}^{n+1}A_i\big(\alpha_i^{-\alpha_i}\big)\cdot B\big(\alpha^{\alpha}\big)$$
$$\times\sum_{\chi\in\widehat{\mathbb{F}_q^\times}}\prod_{i=1}^{n+1}G\big(\overline{A_i\chi}\big)\cdot G\big(B\chi\big)\prod_{i=1}^{n+1}\prod_{b_i=1}^{\alpha_i-1}\frac{G\big(\overline{A_i\chi}\varphi_{\alpha_i}^{b_i}\big)}{G\big(\varphi_{\alpha_i}^{b_i}\big)}\cdot\prod_{b=1}^{\alpha-1}\frac{G\big(B\chi\varphi_{\alpha}^{b}\big)}{G\big(\varphi_{\alpha}^{b}\big)}\chi\left((-1)^{\alpha}\frac{\alpha^{\alpha}}{\alpha_1^{\alpha_1}\dots\alpha_{n+1}^{\alpha_{n+1}}}x\right).$$

This shows the proposition. □

Proposition 3.13 and 3.15 completes the proof of Proposition 3.10.

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NATIONAL CENTER FOR THEORETICAL SCIENCES, MATH. DIVISION,, NATIONAL TAIWAN UNIVERSITY, NO. 1, SEC. 4, ROOSEVELT ROAD, TAIPEI

E-mail address: miyatani@math.i3vi3.xyz